mental results. They then empirically chose a value of $\mu=1.80$ based on the experimental results of Au and found that this same value of μ gives good results for Be. For the potential of (5) to reduce to Molière's representation of the Thomas-Fermi potential $\mu=1$. We have calculated the $1/e$ width corresponding to the experiments of Hanson *et al.* for all three values of μ. The results are shown in Table I where we have compared the $\theta_{1/e}$ of Molière (for $\mu=1.00$) and Nigam *et al.* (for $\mu=1.12$ and $\mu=1.80$) as listed in Table II of Nigam *et al.* Scott[6] has recalculated the results of Nigam *et al.* and obtains values very close to ours. He obtains $\theta_{1/e}=4.05°$ and 3.71° for Au and 4.50° and 4.21° for Be with $\mu=1.12$ and 1.80, respectively, compared to our results of 4.06° and 3.72° for Au and 4.55° and 4.28° for Be. In Table II some numerical values of the first two terms of (16) are given. The fact that the Thomas-Fermi function falls off so slowly with distance and hence is unrealistic for small-angle scattering probably accounts for the lack of agreement with experiment for $\mu=1$. The agreement of our results with those of the Nigam *et al.* theory (as recalculated by Scott) is due to the similarity in functional form between (1) and (5).

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Functional Analysis and Scattering Theory*

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We formulate the nonrelativistic scattering problem as an integral equation with a kernel which is completely continuous for all energies. We then are able to give a rigorous justification for the Fredholm method, quasiparticle method, and, for weak enough interactions, the Born expansion. We also give an explicit lower bound for the radius of convergence of the Born series and of the Born series modified by the introduction of quasiparticles. We furthermore show that all these expansions coverage uniformly in the physical region of energy and momentum transfer.

I. INTRODUCTION

THIS paper is concerned with the application of functional analysis to the problem of scattering of a single nonrelativistic particle by a fixed interaction V. Our purpose when we began this work was to provide a rigorous justification for the "quasiparticle method" presented by one of us in previous papers.[1,2] The sticky point was that the scattering kernel $[W-H_0]^{-1}V$ is not even bounded in the physical scattering region $W\geqq 0$, though it is L^2 for all other W. We overcome this problem here by using a new "symmetrized" kernel[3]

$$V^{\frac{1}{2}}[W-H_0]^{-1}V^{\frac{1}{2}},$$

which is L^2 for all W. (Sec. II and IV.)

Having solved our original problem in this way, we were pleased to find a number of useful by-products:

(1) We give an explicit lower bound on the radius of convergence of the ordinary Born series for all energies. This had previously been done for the bound-state problem[2] but not for the scattering problem (Sec. III). In fact, we give explicit upper bounds on the nth order terms of the Fredholm and Born series (Sec. V), which should be useful for practical calculations.

(2) We do the same for the Born series modified by the introduction of a "quasiparticle," so that it is possible to be certain that the modified Born series converges (Sec. IV).

(3) We show that all these expansions [Fredholm, quasi-Born, and, for weak enough interactions, ordinary Born] converge *uniformly* in the physical region of energy and momentum transfer (Sec. V).

Most of our work is applicable to very general interactions, but we give special attention to the case of a local (not necessarily central) potential $V(\mathbf{r})$, subject

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[1] S. Weinberg, Phys. Rev. **130**, 776 (1963).

[2] S. Weinberg, Phys. Rev. **131**, 440 (1963).

[3] While this paper was being written we received a preprint by F. Coester, Phys. Rev. **133**, B1516 (1964), who uses essentially the same idea. He factors the potential as $V=V^{(+)}V^{(-)}$, and studies the kernel $V^{(-)}[W-H_0]^{-1}V^{(+)}$. The advantage in our choosing $V^{(+)}=V^{(-)}=V^{\frac{1}{2}}$ is that it minimizes the L^2 norm of the scattering kernel, thereby giving a superior lower bound on the radius of convergence of the Born series. The first author to use $V^{\frac{1}{2}}$ symmetrization appears to be J. Schwinger [Proc. Natl. Acad. Sci. U. S. **47**, 122 (1961)], who employed it to study the bound state problem. The general idea of performing similarity transformations on the scattering kernel has also been discussed by L. Brown, D. Fivel, B. W. Lee, and R. Sawyer [Ann. Phys. (N. Y.) **23**, 167 (1963)]. These last authors concentrate on the kernel $[W-H_0]^{-\frac{1}{2}}V[W-H_0]^{-\frac{1}{2}}$; in this connection see also footnote 8 of Ref. 2 and B. Lee, in *Theoretical Physics* [International Atomic Energy Agency, Vienna, 1963], p. 331. However, the kernel $[W-H_0]^{-\frac{1}{2}}V[W-H_0]^{-\frac{1}{2}}$ is not L^2 for $W>0$.

to the conditions:

$$\int d^3r d^3r' \frac{|V(\mathbf{r})||V(\mathbf{r}')|}{|\mathbf{r}-\mathbf{r}'|^2} < \infty\,, \tag{1.1}$$

$$\int d^3r\,|V(\mathbf{r})| < \infty\,. \tag{1.2}$$

These hold if $V(\mathbf{r})$ is $\mathcal{O}(r^{-2+\epsilon})$ for $r \to 0$, and $\mathcal{O}(r^{-3-\epsilon})$ for $r \to \infty$. Condition (1.1) is needed for the symmetrized kernel to be L^2 for all energy, while (1.2) guarantees that all state vectors of interest have finite and uniformly bounded norm.

It is probably straightforward to apply these ideas to the multiparticle scattering problem, by "symmetrizing" the full irreducible connected kernel[4] rather than the Lippmann-Schwinger kernel. We hope to discuss multiparticle problems as well as the relativistic Bethe-Salpeter problem in future articles.

II. THE COMPLETELY CONTINUOUS SCATTERING KERNEL

Let us first recall the difficulties encountered with the usual operator Lippmann-Schwinger equation.

$$T(W) = V + T(W)G_0(W)V = V + VG_0(W)T(W)\,. \tag{2.1}$$

Here V is the interaction, and $G_0(W)$ is the free-particle Green's function at energy W.

$$G_0(W) = [W - H_0]^{-1}. \tag{2.2}$$

In position space[5]

$$\langle \mathbf{r}'|G_0(W)|\mathbf{r}\rangle = \frac{1}{(2\pi)^3}\int d^3p \frac{\exp[i\mathbf{p}\cdot(\mathbf{r}'-\mathbf{r})]}{W-\mathbf{p}^2} = -\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{2.3}$$

with k defined by

$$W = k^2; \quad \mathrm{Im}k > 0. \tag{2.4}$$

We will be particularly (but not exclusively) interested in the case of local V, with

$$\langle \mathbf{r}'|V|\mathbf{r}\rangle = \delta^3(\mathbf{r}'-\mathbf{r})V(\mathbf{r})\,. \tag{2.5}$$

The S matrix at a physical energy $E>0$ is calculated from the formula

$$\langle \mathbf{p}'|S|\mathbf{p}\rangle = \delta^3(\mathbf{p}'-\mathbf{p}) - 2\pi i\delta(\mathbf{p}^2-\mathbf{p}'^2) \times \langle \mathbf{p}'|T(\mathbf{p}^2+i\epsilon)|\mathbf{p}\rangle. \tag{2.6}$$

The "$i\epsilon$" in (2.6) is understood to mean that k in (2.3) is chosen as the limit of $(\mathbf{p}^2+i\epsilon)^{\frac{1}{2}}$ as $\epsilon \to 0+$, i.e.,

$$k = +|\mathbf{p}|\,. \tag{2.7}$$

Equations like (2.1) are perfectly tractable if the kernel is completely continuous, and, in particular, if it is "L^2." The kernel of (2.1) is

$$K(W) = G_0(W)V \tag{2.8}$$

and its L^2 norm is

$$\mathrm{Tr}\{K(W)K^\dagger(W)\} = \mathrm{Tr}\{G_0^\dagger(W)G_0(W)VV^\dagger\} \tag{2.9}$$

or for local potentials

$$\mathrm{Tr}\{K(W)K^\dagger(W)\} = \frac{1}{16\pi^2}\int d^3r \int d^3r' \frac{\exp(-2\,\mathrm{Im}k|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|^2}|V(\mathbf{r})|^2 = \frac{1}{8\pi\,\mathrm{Im}k}\int d^3r\,|V(\mathbf{r})|^2. \tag{2.10}$$

If $V(\mathbf{r})$ is square-integrable, then this trace is finite for W negative (the bound-state region) or W complex, but for $W=\mathbf{p}^2+i\epsilon$, it blows up as ϵ^{-1} for $\epsilon \to 0$. Since $K(W)$ is not L^2 in the scattering region (in fact it is an unbounded operator), it was necessary in earlier work to appeal to an imperfectly rigorous analytic continuation from complex to real W to justify the treatment of $K(\mathbf{p}^2+i\epsilon)$ as if it were completely continuous. Another difficulty with the usual formulation of scattering theory is that the matrix element $\langle \mathbf{p}'|T|\mathbf{p}\rangle$ is taken between continuum states whose wave functions $e^{i\mathbf{p}\cdot\mathbf{r}}$, $e^{i\mathbf{p}'\cdot\mathbf{r}}$ are not normalizable, and hence not in Hilbert space.

Instead of $K(W)$, let us define a new "symmetrized" kernel[3]

$$\tilde{K}(W) = V^{\frac{1}{2}}G_0(W)V^{\frac{1}{2}}. \tag{2.11}$$

For local potentials this gives in position space

$$\langle \mathbf{r}'|\tilde{K}(W)|\mathbf{r}\rangle = -V^{\frac{1}{2}}(\mathbf{r}')V^{\frac{1}{2}}(\mathbf{r})\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{4\pi|\mathbf{r}-\mathbf{r}'|}\,. \tag{2.12}$$

Either sign can be taken for $V^{\frac{1}{2}}(\mathbf{r}')$; the fact that $V^{\frac{1}{2}}(\mathbf{r})$ is imaginary where $V(\mathbf{r})$ is negative will cause no trouble.

It is easy to see that (2.1) has the formal solution

$$T(W) = V^{\frac{1}{2}}[1-\tilde{K}(W)]^{-1}V^{\frac{1}{2}} \tag{2.13}$$

giving the S matrix

$$\langle \mathbf{p}'|S|\mathbf{p}\rangle = \delta^3(\mathbf{p}'-\mathbf{p}) - 2\pi i\delta(\mathbf{p}^2-\mathbf{p}'^2) \times \langle \mathbf{p}'|V^{\frac{1}{2}}[1-\tilde{K}(\mathbf{p}^2+i\epsilon)]^{-1}V^{\frac{1}{2}}|\mathbf{p}\rangle. \tag{2.14}$$

The reader may rest assured that the ugly operator $V^{\frac{1}{2}}$ will not appear in actual calculations of the S matrix. For instance, the series expansion of (2.13) in powers of $\tilde{K}(W)$ is

$$T(W) = V^{\frac{1}{2}}[1+\tilde{K}(W)+\tilde{K}(W)^2+\cdots]V^{\frac{1}{2}} = V+VG_0(W)V+VG_0(W)VG_0(W)V+\cdots \tag{2.15}$$

and this is just the ordinary Born series. We will see in Secs. IV and V that the Fredholm and quasiparticle

[4] S. Weinberg, Phys. Rev. **133**, B232 (1964).
[5] We use units with $\hbar = 2m = 1$.

series for (2.13) are the same as they were before the symmetrization of the kernel; the general reason is that $K(W)$ and $\tilde{K}(W)$ are related by a formal similarity transformation.

$$\tilde{K}(W)V^{\frac{1}{2}}=V^{\frac{1}{2}}K(W). \tag{2.16}$$

There are three great advantages in using Eq. (2.14) as the starting point of scattering theory:

(α) The kernel $\tilde{K}(W)$ is L^2 for decent interactions. Its L^2 norm is

$$\begin{aligned}\tilde{\tau}(W)&\equiv\mathrm{Tr}\{\tilde{K}(W)\tilde{K}^\dagger(W)\}\\&=\mathrm{Tr}\{V^{\frac{1}{2}\dagger}V^{\frac{1}{2}}G_0(W)V^{\frac{1}{2}}V^{\frac{1}{2}\dagger}G_0(W^*)\}.\end{aligned} \tag{2.17}$$

In contrast with (2.9), $\tilde{\tau}(W)$ does *not* generally diverge as W approaches the positive real axis, since the dangerous denominators $[W-H_0]$ and $[W^*-H_0]$ are kept apart. For example, if V is a local potential, (2.17) gives

$$\tilde{\tau}(W)=\frac{1}{16\pi^2}\int d^3r d^3r' \times\frac{|V(\mathbf{r})||V(\mathbf{r}')|\exp(-2\,\mathrm{Im}k|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|^2}, \tag{2.18}$$

and for $W=p^2+i\epsilon$, this takes the W-independent value

$$\tilde{\tau}=\frac{1}{16\pi^2}\int d^3r d^3r'\frac{|V(\mathbf{r})||V(\mathbf{r}')|}{|\mathbf{r}-\mathbf{r}'|^2}. \tag{2.19}$$

This integral is obviously finite for reasonable $V(\mathbf{r})$, like the Yukawa potential (but, unfortunately, not for the Coulomb potential). Specific examples will be worked out in Sec. III.

(β) In the bound-state region $W<0$, both $K(W)$ and $\tilde{K}(W)$ may be L^2, but $\tilde{K}(W)$ always has the smaller L^2 norm. Hence the condition that $\tilde{\tau}(W)$ be finite is weaker than our previous condition that $V(\mathbf{r})$ be square-integrable. Also the condition $\tilde{\tau}(-B)>1$ gives a better upper bound on the binding energy B of bound states than previously given in Eq. (132) of Ref. 2. [Schwinger[3] has shown that the number of bound states below $-B$ is less than $\tilde{\tau}(-B)$.]

(γ) The S matrix is a matrix element of $[1-\tilde{K}(W)]^{-1}$ between states that actually lie in Hilbert space, since the state vector $V^{\frac{1}{2}}|\mathbf{p}\rangle$ appearing in (2.14) has norm

$$\|V^{\frac{1}{2}}|\mathbf{p}\rangle\|^2=\langle\mathbf{p}|V^{\frac{1}{2}\dagger}V^{\frac{1}{2}}|\mathbf{p}\rangle \tag{2.20}$$

or for local potentials

$$\|V^{\frac{1}{2}}|\mathbf{p}\rangle\|^2=\frac{1}{(2\pi)^3}\int d^3\mathbf{r}|V(\mathbf{r})|. \tag{2.21}$$

This norm exists for decent short-range interactions like the Yukawa potential.

These two facts, that $\tilde{K}(W)$ is L^2 and $V^{\frac{1}{2}}|\mathbf{p}\rangle$ is normalizable, allow us without further ado to apply the standard lore of functional analysis[6] to the operator $[1-\tilde{K}(W)]^{-1}$ and its matrix elements. In particular, it follows immediately that the S matrix for interaction λV is a meromorphic function of the coupling constant λ, for all real or complex W. Its poles are at the λ values

$$\lambda=\eta_\nu^{-1}(W), \tag{2.22}$$

where the $\eta_\nu(W)$ are the eigenvalues of $\tilde{K}(W)$.

$$\tilde{K}(W)|\tilde{\Psi}_\nu(W)\rangle=\eta_\nu(W)|\tilde{\Psi}_\nu(W)\rangle, \tag{2.23}$$

or for local potentials

$$\begin{aligned}-\frac{V^{\frac{1}{2}}(\mathbf{r})}{4\pi}\int d^3r'\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}V^{\frac{1}{2}}(\mathbf{r}')\tilde{\psi}_\nu(\mathbf{r}';W)\\=\eta_\nu(W)\tilde{\psi}_\nu(\mathbf{r};W),\\ \tilde{\psi}_\nu(\mathbf{r};W)\equiv\langle\mathbf{r}|\tilde{\Psi}_\nu(W)\rangle.\end{aligned} \tag{2.24}$$

The eigenvector $|\tilde{\Psi}_\nu(W)\rangle$ is, of course, understood to lie in Hilbert space, i.e., to have finite norm

$$\|\tilde{\Psi}_\nu(W)\|^2\equiv\int|\tilde{\psi}_\nu(\mathbf{r};W)|^2d^3r<\infty. \tag{2.25}$$

It may be noted that the $\eta_\nu(W)$ can also be described as the eigenvalues of the original kernel $K(W)$, since (2.23) can be written

$$|\tilde{\Psi}_\nu(W)\rangle=V^{\frac{1}{2}}|\Psi_\nu(W)\rangle, \tag{2.26}$$

$$K(W)|\Psi_\nu(W)\rangle=\eta_\nu(W)|\Psi_\nu(W)\rangle. \tag{2.27}$$

However, the normalizability condition (2.25) now reads

$$\|V^{\frac{1}{2}}|\Psi_\nu(W)\rangle\|<\infty, \tag{2.28}$$

or for local potentials

$$\int d^3r|V(\mathbf{r})||\psi_\nu(\mathbf{r};W)|^2<\infty. \tag{2.29}$$

It is easy to see that $\psi_\nu(r;W)$ behaves like e^{ikr} as $r\to\infty$; hence, for real k the integrating factor $|V(\mathbf{r})|$ in (2.29) is indispensable to obtain a finite norm.

The implications of (α) and (γ) for the various series expansions of the S matrix will be considered in Secs. IV and V.

III. RADIUS OF CONVERGENCE

We now turn to a calculation of the L^2-norm $\tilde{\tau}(W)$, given by (2.18). This is a matter of some practical importance, because all eigenvalues $\eta_\nu(W)$ of $\tilde{K}(W)$ are subject to the inequality

$$|\eta_\nu(W)|^2\leqq\tilde{\tau}(W). \tag{3.1}$$

The radius of convergence $R(W)$ of the Born series for an interaction λV at energy W is equal to the smallest $|\eta_\nu(W)|^{-1}$, so (3.1) provides a lower bound on the radius of convergence.

$$R(W)\geqq\tilde{\tau}^{-\frac{1}{2}}(W). \tag{3.2}$$

[6] For a survey and additional references, see Appendix A of Ref. 4.

This inequality is particularly useful because we shall find that in practice $\tilde{\tau}^{-\frac{1}{2}}(W)$ is only slightly less than the exact $R(W)$.

Instead of calculating (2.18) directly, it is easier to return to (2.17), and write it in momentum space as

$$\tilde{\tau}(W)=\int d^3p d^3p' \frac{|U(\mathbf{p}-\mathbf{p}')|^2}{(W-\mathbf{p}^2)(W^*-\mathbf{p}'^2)} \tag{3.3}$$

$$=\int d^3q |U(\mathbf{q})|^2 \times \int d^3\mathbf{p} \frac{1}{(W-\mathbf{p}^2)[W^*-(\mathbf{p}-\mathbf{q})^2]}, \tag{3.4}$$

where

$$U(\mathbf{q})=(2\pi)^{-3}\int d^3r |V(\mathbf{r})| e^{i\mathbf{q}\cdot\mathbf{r}}. \tag{3.5}$$

The $\mathbf{p}$ integral in (3.4) can be done by Feynman's method, yielding

$$\tilde{\tau}(W)=2\pi^2\int d^3\mathbf{q} |U(\mathbf{q})|^2 \arctan\left(\frac{q}{2\,\mathrm{Im}k}\right)\Big/ q. \tag{3.6}$$

We see again that $\tilde{\tau}(W)$ depends only on $\mathrm{Im}(W)^{\frac{1}{2}}$, and that for $W>0$ it becomes W-independent.

$$\tilde{\tau}=\pi^3\int d^3\mathbf{q} |U(\mathbf{q})|^2/q. \tag{3.7}$$

For all complex or negative W we have

$$\tilde{\tau}(W)\leqq\tilde{\tau}, \tag{3.8}$$

so $\tilde{\tau}^{-\frac{1}{2}}$ is a lower bound on the radius of convergence for all W.

We are assuming that (2.21) converges, so $U(\mathbf{p})$ is finite for all $\mathbf{p}$. The integral (3.7) for $\tilde{\tau}$ will then exist if, as $|\mathbf{p}|\to\infty$,

$$U(\mathbf{p})=\mathcal{O}(|p|^{-1-\epsilon}) \quad (\text{any } \epsilon>0), \tag{3.9}$$

and this will hold if, as $|\mathbf{r}|\to 0$,

$$V(\mathbf{r})=\mathcal{O}(|\mathbf{r}|^{-2+\epsilon}) \quad (\text{any } \epsilon>0). \tag{3.10}$$

The condition that (2.21) converges will be satisfied if $V(\mathbf{r})$ is finite for all finite $\mathbf{r}$, satisfies (3.10), and if, as $|\mathbf{r}|\to\infty$,

$$V(\mathbf{r})=\mathcal{O}(|\mathbf{r}|^{-3-\epsilon}) \quad (\text{any}\epsilon>0). \tag{3.11}$$

We will consider two particular examples, the Yukawa potential

$$\lambda V(r)=\lambda\mu e^{-\mu r}/r \quad (\mathrm{Y}) \tag{3.12}$$

and the exponential potential

$$\lambda V(r)=\lambda\mu^2 e^{-\mu r} \quad (\mathrm{E}). \tag{3.13}$$

In these cases (3.5) gives, respectively,

$$U(\mathbf{p})=\mu/2\pi^2(p^2+\mu^2) \quad (\mathrm{Y}), \tag{3.14}$$

$$U(\mathbf{p})=\mu^3/\pi^2(p^2+\mu^2)^2 \quad (\mathrm{E}), \tag{3.15}$$

so (3.7) is

$$\tilde{\tau}=\tfrac{1}{2} \quad (\mathrm{Y}), \tag{3.16}$$

$$\tilde{\tau}=\tfrac{2}{3} \quad (\mathrm{E}). \tag{3.17}$$

We conclude that the Born series converges for all W if

$$|\lambda|\leqq\sqrt{2}=1.414 \quad (\mathrm{Y}), \tag{3.18}$$

$$|\lambda|\leqq(\tfrac{3}{2})^{\frac{1}{2}}=1.225 \quad (\mathrm{E}). \tag{3.19}$$

These results may be compared with the known[7] exact radii of convergence (at $W=0$, where they are smallest).

$$R(0)=1.680 \quad (\mathrm{Y}), \tag{3.20}$$

$$R(0)=1.446 \quad (\mathrm{E}). \tag{3.21}$$

In both cases $\tilde{\tau}^{-\frac{1}{2}}$ is only about 15% smaller than the true radius of convergence.

To see why $\tilde{\tau}^{-\frac{1}{2}}$ is so close to $R(0)$, it will be illuminating to consider another example. Let V be an Hermitian separable interaction

$$\lambda V=\lambda|a\rangle\langle a|, \qquad \langle a|a\rangle=1. \tag{3.22}$$

The symmetrized kernel is proportional to V.

$$\tilde{K}(W)=\langle a|G_0(W)|a\rangle|a\rangle\langle a|. \tag{3.23}$$

Obviously, $\langle a|G_0(W)|a\rangle$ is the greatest eigenvalue of $\tilde{K}(W)$, and furthermore

$$\tilde{\tau}(W)=|\langle a|G_0(W)|a\rangle|^2, \tag{3.24}$$

so $\tilde{\tau}^{-\frac{1}{2}}(W)$ is the exact radius of convergence for (3.22). The close agreement between $\tilde{\tau}^{-\frac{1}{2}}$ and $R(0)$ for the Yukawa and exponential potentials can therefore be traced to the fact that such potentials are effectively separable at low energies,[8] in the sense that one of the eigenvalues $\eta_\nu(W)$ is much larger than all the others.

For local central potentials the radius of convergence in the lth partial wave is greater than $\tilde{\tau}_l^{-\frac{1}{2}}(W)$, where $\tilde{\tau}_l(W)$ is the l-wave L^2 norm.

$$\tilde{\tau}_l(W)=|k|^2\int_0^\infty r^2|V(r)|dr \times\int_0^\infty r'^2|V(r')|dr'|j_l(kr_<)h_l^{(1)}(kr_>)|^2. \tag{3.25}$$

This gives an even better lower bound on the radius of convergence, since each $\tau_l(W)$ is less than $\tau(W)$; in fact,

$$\tilde{\tau}(W)=\sum_{l=0}^{\infty}(2l+1)\tilde{\tau}_l(W). \tag{3.26}$$

[Note that $\tilde{\tau}_l(W)$ depends on W even for W real, though

[7] The Yukawa value is from L. Hulthén and K. V. Laurikainen, Rev. Mod. Phys. **23**, 1 (1951), and the exponential value is from T-Y. Wu and T. Ohmura, *Quantum Theory of Scattering* [Prentice-Hall, Englewood Cliffs, New Jersey, 1962], p. 80.

[8] M. Scadron and S. Weinberg, Phys. Rev. **133**, B1589 (1964).

$\tilde{\tau}(W)$ does not.] At zero energy (3.25) is

$$\tilde{\tau}_l(0)=\frac{1}{(2l+1)^2}\int_0^\infty |V(r)|dr\int_0^\infty |V(r')|dr' r_<^{2l+2}r_>^{-2l}$$
$$=\frac{2}{(2l+1)^2}\int_0^1 x^{2l+2}\int_0^\infty |V(r)||V(xr)|r^3dr. \quad (3.27)$$

For the Yukawa and exponential potentials [(3.12) and (3.13)], Eq. (3.27) gives

$$\tilde{\tau}_l(0)=\frac{2}{(2l+1)^2}\int_0^1 \frac{x^{2l+1}}{(1+x)^2}dx \quad (Y), \quad (3.28)$$

$$\tilde{\tau}_l(0)=\frac{12}{(2l+1)^2}\int_0^1 \frac{x^{2l+2}}{(1+x)^4}dx \quad (E). \quad (3.29)$$

For S waves this gives

$$\tilde{\tau}_0(0)=2\ln 2-1; \quad \tilde{\tau}_0^{-\frac{1}{2}}(0)=1.609 \quad (Y), \quad (3.30)$$

$$\tilde{\tau}_0(0)=\tfrac{1}{2}; \quad \tilde{\tau}_0^{-\frac{1}{2}}(0)=1.414 \quad (E). \quad (3.31)$$

The exact radii of convergence[7] are, respectively, 1.680 and 1.446, just a few percent above $\tilde{\tau}_0^{-\frac{1}{2}}(0)$. For p waves (3.28) and (3.29) give

$$\tilde{\tau}_0(0)=\frac{2}{3}\ln 2-\frac{4}{9}; \quad \tilde{\tau}_0^{-\frac{1}{2}}(0)=7.533 \quad (Y), \quad (3.32)$$

$$\tilde{\tau}_0(0)=\frac{67}{18}-\frac{16}{3}\ln 2; \quad \tilde{\tau}_0^{-\frac{1}{2}}(0)=6.239 \quad (E). \quad (3.33)$$

The value (3.32) compares favorably with the known[9] zero energy p-wave Yukawa radius of convergence, $R(0)\cong 9.1$.

IV. THE QUASIPARTICLE METHOD

The quasiparticle method[2] may be described for our present purposes as the replacement of $\tilde{K}(W)$ by a kernel $K_Q(W)$, which differs by a term of finite rank.

$$K_Q(W)=\tilde{K}(W)-\sum_s V^{\frac{1}{2}}|s\rangle\langle\bar{s}|V^{\frac{1}{2}}. \quad (4.1)$$

Here $|s\rangle$ and $\langle\bar{s}|$ are a finite set of state vectors (perhaps W-dependent) that can be chosen as we like. It is easy to show that

$$T(W)=T_Q(W)+\sum_{s,s'}T_Q(W)|s\rangle \times\Delta_{ss'}(W)\langle\bar{s}'|T_Q(W), \quad (4.2)$$

$$[\Delta^{-1}(W)]_{ss'}\equiv\delta_{ss'}-\langle\bar{s}'|T_Q(W)|s\rangle, \quad (4.3)$$

$$T_Q(W)\equiv V^{\frac{1}{2}}[1-K_Q(W)]^{-1}V^{\frac{1}{2}}. \quad (4.4)$$

The point is to choose the $|s\rangle$ and $\langle\bar{s}|$ so that (4.4) may be calculated by expansion in $K_Q(W)$.

$$T_Q(W)=V^{\frac{1}{2}}[1+K_Q(W)+K_Q(W)^2+\cdots]V^{\frac{1}{2}}, \quad (4.5)$$
$$=V+\{VG_0(W)V-\sum_s V|s\rangle\langle\bar{s}|V\}+\cdots. \quad (4.6)$$

This is always possible because $\tilde{K}(W)$ is completely continuous for all W, and hence may always be approximated uniformly by a kernel of finite rank; that is, it is possible for all W to choose $|s\rangle$ and $\langle\bar{s}|$ so that $\|K_Q(W)\|$ is as small as we like, and in particular, so that $\|K_Q(W)\|<1$. In the original work on the quasiparticle method, $K(W)$ was used instead of $\tilde{K}(W)$, and it was necessary to resort to hand-waving to discuss real scattering energies, for which $K(W)$ is not completely continuous. [It should perhaps be emphasized that (4.1)–(4.6) are just a rewritten version of the original quasiparticle method, but the method is now rigorously justified.]

How should we choose the $|s\rangle$ and $\langle\bar{s}|$? Our previous answer[2] was that we should try to reduce all eigenvalues $\eta_s(W)$ with $|\eta_s(W)|>1$ to zero, and leave all the other $\eta_\nu(W)$ unchanged. For instance, if only one eigenvalue $\eta_1(W)$ lies outside the unit circle, then the "ideal" choice according to this prescription would be

$$|1\rangle\propto|\psi_1(W)\rangle; \quad \langle\bar{1}|\propto\langle\psi_1(W^*)| \quad (4.7)$$

with normalization and phase chosen so that

$$\langle\bar{1}|V|1\rangle=\eta_1(W). \quad (4.8)$$

We have already done some practical calculations,[8] choosing

$$|1\rangle=G_0(W)V|\Gamma(W)\rangle; \quad \langle\bar{1}|=\langle\bar{\Gamma}(W)|, \quad (4.9)$$

where $|\Gamma(W)\rangle$ and $\langle\bar{\Gamma}(W)|$ were an educated guess at $|\psi_1(W)\rangle$ and $\langle\psi_1(W^*)|$, with normalization

$$\langle\bar{\Gamma}(W)|V|\Gamma(W)\rangle=1. \quad (4.10)$$

[This would agree with (4.8) if $|\Gamma\rangle$ and $\langle\bar{\Gamma}|$ were exact eigenfunctions.] The results obtained in lowest order were excellent, but we were then unable to say with mathematical certainty that we had succeeded in making the series (4.5) converge.

However, we *can* be certain that (4.5) converges if

$$\tau_Q(W)<1, \quad (4.11)$$

where $\tau_Q(W)$ is the L^2 norm of $K_Q(W)$.

$$\tau_Q(W)=\mathrm{Tr}\{K_Q(W)K_Q(W)^\dagger\}$$
$$=\tilde{\tau}(W)-2\,\mathrm{Re}\sum_s\langle\bar{s}|V^{\frac{1}{2}}V^{\frac{1}{2}\dagger}G_0(W^*)V^{\frac{1}{2}\dagger}V^{\frac{1}{2}}|s\rangle$$
$$+\sum_{s,s'}\langle s'|V^{\frac{1}{2}\dagger}V^{\frac{1}{2}}|s\rangle\langle\bar{s}|V^{\frac{1}{2}}V^{\frac{1}{2}\dagger}|\bar{s}'\rangle. \quad (4.12)$$

With a good guess at the form of $|s\rangle$ and $\langle\bar{s}|$, it should be easy to adjust their normalization and phase so that (4.11) is satisfied.

[9] This can be obtained from the graph on p. 481, C. Lovelace and D. Masson, Nuovo Cimento **26**, 472 (1962).

V. UNIFORM CONVERGENCE OF THE RESOLVENT EXPANSIONS

Since $\tilde{K}(W)$ is an L^2 kernel for all W, we are rigorously justified in evaluating the resolvent $[1-\tilde{K}(W)]^{-1}$ by the modified Fredholm formulas.[10]

$$[1-\tilde{K}(W)]^{-1}=1+\tilde{N}(W)/\tilde{D}(W), \tag{5.1}$$

$$\tilde{N}(W)=\sum_{n=0}^{\infty}\tilde{N}_n(W), \tag{5.2}$$

$$\tilde{D}(W)=\sum_{n=0}^{\infty}\tilde{D}_n(W), \tag{5.3}$$

the operators $\tilde{N}_n(W)$ and the functions $\tilde{D}_n(W)$ being given by the recursion relations

$$\tilde{D}_{n+1}(W)=-\frac{1}{n+1}\operatorname{Tr}\{\tilde{N}_n(W)-\tilde{D}_n(W)\tilde{K}(W)\}, \tag{5.4}$$

$$\tilde{N}_{n+1}(W)=[\tilde{D}_{n+1}(W)+\tilde{N}_n(W)]\tilde{K}(W), \tag{5.5}$$

$$\tilde{D}_0(W)=1;\quad \tilde{N}_0(W)=\tilde{K}(W). \tag{5.6}$$

It is easy to see that the Fredholm numerator and denominator for $\tilde{K}(W)$ are related to the numerator $N(W)$ and denominator $D(W)$ for the original kernel $K(W)$, by

$$\tilde{N}(W)V^{\frac{1}{2}}=V^{\frac{1}{2}}N(W), \tag{5.7}$$

$$\tilde{D}(W)=D(W) \tag{5.8}$$

so that the T operator (2.13) is

$$T(W)=V+V^{\frac{1}{2}}\tilde{N}(W)V^{\frac{1}{2}}/\tilde{D}(W) =V+VN(W)/D(W). \tag{5.9}$$

Hence, symmetrization affords an easy and yet rigorous proof for the applicability of the usual Fredholm method.

But symmetrization does even more. It is well known that the Fredholm series for $\tilde{N}(W)$ converges uniformly in Hilbert space because the L^2 norm of the nth term is bounded by[10]

$$\operatorname{Tr}\{\tilde{N}_n(W)\tilde{N}_n(W)^\dagger\}\leqq n^{-n}[e\tilde{\tau}(W)]^{n+1}. \tag{5.10}$$

It follows from (5.9), (5.10), and (2.21) that for local potentials the $n+1$th term in the numerator of the T matrix is bounded by

$$|\langle\mathbf{p}'|V^{\frac{1}{2}}\tilde{N}_n(W)V^{\frac{1}{2}}|\mathbf{p}\rangle| \leqq n^{-n/2}[e\tilde{\tau}(W)]^{(n+1)/2}(2\pi)^{-3}\int d^3r\,|V(\mathbf{r})|. \tag{5.11}$$

Also, the nth term of the denominator is bounded by[10]

$$|\tilde{D}_n(W)|\leqq n^{-n/2}[e\tilde{\tau}(W)]^{(n+1)/2}. \tag{5.12}$$

[10] F. Smithies, *Integral Equations* (Cambridge University Press, New York, 1958).

Using (3.8), let us write these inequalities as

$$|\langle\mathbf{p}'|V^{\frac{1}{2}}\tilde{N}_n(W)V^{\frac{1}{2}}|\mathbf{p}\rangle| \leqq n^{-n/2}[e\tilde{\tau}]^{(n+1)/2}(2\pi)^{-3}\int d^3r\,|V(\mathbf{r})|, \tag{5.13}$$

$$|\tilde{D}_n(W)|\leqq n^{-n/2}[e\tilde{\tau}]^{(n+1)/2}. \tag{5.14}$$

So we see that *the Fredholm expansions converge uniformly in* $\mathbf{p}'$, $\mathbf{p}$, *and* W. This result has been obtained before,[11] but by a tortuous method and under strongly restrictive conditions on the potential.

Uniform convergence is not restricted to the Fredholm expansions. Let us suppose, for example, that the interaction is sufficiently weak so that

$$\tilde{\tau}<1. \tag{5.15}$$

Then the ordinary Born series converges uniformly in Hilbert space, because

$$\|[1-\tilde{K}(W)]^{-1}-\sum_{n=0}^{N}\tilde{K}(W)^n\|\leqq\sum_{n=N+1}^{\infty}\|\tilde{K}(W)\|^n \leqq\tilde{\tau}^{(N+1)/2}/(1-\tilde{\tau}^{\frac{1}{2}}). \tag{5.16}$$

Using (2.21) again gives

$$|\langle\mathbf{p}'|T(W)|\mathbf{p}\rangle -\langle\mathbf{p}'|V+VG_0(W)V+\cdots V(G_0(W)V)^N|\mathbf{p}\rangle| \leqq\frac{\tilde{\tau}^{(N+1)/2}}{(1-\tilde{\tau}^{\frac{1}{2}})(2\pi)^3}\int d^3r\,|V(\mathbf{r})|, \tag{5.17}$$

so (5.15) ensures that the ordinary Born series converges uniformly in $\mathbf{p}'$, $\mathbf{p}$, and W. [The series actually converges uniformly whenever it converges, even if $\tilde{\tau}>1$.]

In the same way, the modified Born series (4.5) for $T_Q(W)$ will converge uniformly in $\mathbf{p}$, $\mathbf{p}'$ and in W as long as $\tau_Q(W)<1$.

These uniformity properties are useful, both for practical calculations, and for the study of analyticity properties[12] of the S matrix.

Note added in proof. After this article was submitted for publication, we were informed that the idea of $V^{1/2}$ symmetrization had been used to study the scattering problem by A. Grossman and T. Wu, J. Math. Phys. **2**, 710 (1961), and by K. Meetz, J. Math. Phys. **3**, 690 (1962). (We are grateful to F. J. Dyson and K. Chadan for these references.) Our emphasis in this article is rather different, and most of our applications are new. Dyson also points out that the Banach space approach of W. Hunziker [Helv. Phys. Acta **34**, 593 (1961)] may suit scattering problems more naturally than the Hilbert space approach described here.

[11] R. Jost and A. Pais, Phys. Rev. **82**, 840 (1951); N. Khuri, Phys. Rev. **107**, 1148 (1957).

[12] R. Blankenbecler, M. L. Goldberger, N. Khuri, and S. B. Treiman, Ann. Phys. (N. Y.) **10**, 62 (1960).